RECENT PUBLICATIONS.

REVIEWS.

History of the Theory of Numbers.

History of the Theory of Numbers. Volume II:[1] Diophantine Analysis. By LEONARD EUGENE DICKSON. Carnegie Institution of Washington, 1920, publication number 256, Vol. II. 26 + 803 pages. Price, paper, $7.50; cloth, $8.00.

In one respect at least Diophantine Analysis is probably unique in the history of mathematics. Perhaps no other division of the whole field has at the same time furnished the subject of such numerous investigations for so many generations of mathematicians and yet has received so little systematic development. Some pleasing chapters are to be found in an exposition of the subject; and a few of the most beautiful theorems in mathematics belong to it. But trivial problems have been too often treated; and fragmentary and incomplete results are to be found dispersed throughout nearly the whole literature.

The nature of the subject has made it an easy prey to this evil. Two Diophantine equations which are much alike in external form may be totally different as regards the essential characteristics of their theory. One may be easy to treat and the other may be exceedingly difficult. A mediocre investigator can always find for himself some of these easy problems; and too frequently he has been willing to publish unimportant results. Some of the papers are of the character which would be produced if one who had failed in larger problems set out to find something of difficulty proportionate to his strength and then published whatever he found. Other papers are at the opposite extreme and have required for their production a command of a wide range of methods and the deepest insight on the part of the investigator. If this judgment concerning the less fortunate investigator seems to the reader to be ungenerous or even harsh he would probably have more sympathy with it after proceeding laboriously through some hundreds or thousands of pages of the more trivial articles and notes.

The immense number of disjointed elements brought to definite notice by a systematic and complete account such as that of the volume under review impels one to believe that the time has come for a change in the methods of developing the Diophantine Analysis. Diophantus himself and many of his followers have been content with special solutions of their problems obtained under restrictive hypotheses which are employed for no other reason than that they simplify the analysis. Such papers have at least the value of showing that the equations treated are not impossible. Moreover, these partial investigations have made clear the essential character of several types of equations which repeatedly recur as auxiliary to the solution of other problems.

But there seems to be no further need of the disjointed detail which is derived

[1] For a review of volume I see this MONTHLY, *1919*, 396–403.

by tentative methods of no wide range and which serves merely to cumber the literature with more uninteresting detail and separated fact answering no real question. "Since there already exist too many papers on Diophantine Analysis which give only special solutions, it is hoped that all devotees of this subject will in future refrain from publication until they obtain general theorems on the problem attacked if not a complete solution of it. Only in this way will the subject be able to retain its proper position by the side of other virile branches of mathematics" (p. xx). "A. Hurwitz's complete discussion (p. 697) of the positive integral solutions of $x_1^2 + \cdots + x_n^2 = xx_1 \cdots x_n$ furnishes a model for thoroughness which may well be imitated by writers on Diophantine equations, too many of whom seem to be content with a special solution of their problems" (p. xvii).

Ideas rather than computations are needed in this field. Some organizing force to bring order out of chaos is essential. If it is not supplied and if the accumulation of small detail continues there is danger that Diophantine Analysis will become an ungainly monster and a reproach to those who cultivate its acquaintance.

In criticizing the subject for the predominance of isolated problems one must not overlook the fact that unattached results of some sorts are of real use, namely, those which answer a real question. It seems not to be known, for instance, whether the sum of five fifth powers can itself be a fifth power just as there was a time when it was similarly unknown whether the sum of four fourth powers could itself be a fourth power. It was worth while to have the latter question answered affirmatively (p. 652); and it would likewise be of value to have the former (or any of its generalizations, pp. 648, 682) answered affirmatively by means of an example (if such exists).

Every one interested in Diophantine Analysis must have observed how it ties the ages together for the student of mathematics. A similar honor belongs to astronomy, logic, and geometry. In the case of Diophantine Analysis this connection has been maintained primarily by a continued interest in isolated Diophantine problems. In logic there has been an accumulating discussion of the method of reasoning. In astronomy observations have been made from time immemorial and these in more recent generations have been reduced to order in the theories of Celestial Mechanics. In geometry an elegant and satisfying statement was made by Euclid in a form to serve as a model even down to our own time; and the moderns have extended the subject into wide ramifications. But Diophantine Analysis has existed principally in a long chain of isolated problems and results. This, though the general fact, fails to be true of some topics of the subject.

A good example of the latter is that afforded by the remarkable theorem that every positive integer is a sum of four squares. Diophantus employed sums of four squares in three problems without naming any condition on a number in order that it shall be a sum of four squares while he did give such necessary conditions in similar cases for representations as sums of two or of three squares. On account of these facts Bachet (in 1621) and Fermat (in 1636) expressed the

judgment that Diophantus probably had a knowledge of the theorem that every positive integer is a sum of four squares; and the latter stated that he possessed a proof by infinite descent. For more than forty years Euler gave repeated attention to the theorem. He converted it into equivalent forms; but he was constantly baffled in his attempt to find a proof. "Not until twenty years after he began the study of the theorem did he publish in 1751 some important facts bearing on it, including his formula which expresses the product of two sums of four squares as such a sum" (p. x).

Lagrange, acknowledging his indebtedness to Euler's paper, published the first proof of the theorem in 1772; but the method is rather complicated. A much more elegant proof was given by Euler in the following year, a proof which has not been improved upon to the present time, though several others have been offered.

But the history of the theorem is not closed with the discovery of these proofs. The inevitable question arises as to the number of representations of a given integer n as a sum of four squares; and this was answered by Jacobi in a remarkable theorem (p. x) obtained by comparing two infinite series for the same elliptic function. Several elementary proofs of the theorem of Jacobi have been given (p. x), one of them as late as 1914, while a proof by means of theta functions was given in 1915.

Here we have, not isolated facts, but a general theorem of great beauty and interest which has served as an intellectual bond among mathematicians of several centuries. A similar connection is afforded, perhaps in an even more remarkable manner, by the theorem that every prime of the form $4n + 1$ is a sum of two squares and by its generalizations.

The book contains reports on more than five thousand writings. The method and point of view of the author in preparing these is briefly indicated in the following words from page xx of the preface: "While many of these papers are of minor importance, the aim has been to give an exhaustive account of the literature on the subject rather than a selective account reflecting the author's imperfect views as to relative importance. This work is intended as a source book not merely for the fastidious professional mathematician, but also for the larger number of amateurs who find endless fascination for the 'queen of the sciences,' whose rule began centuries ago and has continued without interruption to the present."

The table of contents contains an excellent and convenient classification of Diophantine problems and equations (with references to the parts of the text in which they are treated). A preface of twenty pages gives a most valuable outline of material contained in the whole volume and numerous illuminating remarks concerning the history of Diophantine Analysis. This introductory matter in the second volume is better prepared than the corresponding matter in the first volume. In fact, it is to be said that the author's experience in the preparation of volume I has been useful to him in the way of leading to several improvements in volume II. While the former excited our admiration for its remarkable excellences, the latter renews it and makes it keener.

The extent of the volume is too vast for the reviewer to undertake a summary (rendered unnecessary by the preface). Attention will be called merely to a few outstanding features of the book and its general subject matter.

The author believes that his chapter XXIII, on equations of degree greater than 4, will be more useful than any other in the volume since it contains reports on papers which offer general methods of attacking Diophantine equations; the principal methods referred to are mentioned on page xvii of the preface. A high degree of accuracy for chapters III, XXI–XXVI was especially desired since it is thought that they are the ones which will be most frequently consulted. They deal in order with the following topics: partitions; equations of degree three; equations of degree four; equations of degree n; sets of integers with equal sums of like powers; Waring's problem and related results; Fermat's last theorem (with certain closely related matters). Each of these chapters was checked with especial care by the author or by some other mathematician who gave especial attention to the single chapter.

After speaking of the inexhaustible store of interesting truths presented to us by the higher arithmetic and of the wholly unexpected ties which are often discovered among them, Gauss (as quoted in Moritz, *Memorabilia Mathematica*, p. 272) proceeds to add: "A great part of its theories derives an additional charm from the peculiarity that important propositions, with the impress of simplicity upon them, are often easily discoverable by induction, and yet are of so profound a character that we can not find their demonstration till after many vain attempts; and even then, when we do succeed, it is often by some tedious and artificial process, while the simpler methods may long remain concealed."

Several examples of the contrast between the ease with which empirical theorems are discovered and the difficulty attending a complete proof are afforded by Diophantine Analysis. The theorem that every positive integer may be represented as the sum of four squares is an interesting one whose history is instructive (see page x). A simpler case with a shorter history is that of the theorem that every prime of the form $4n + 1$ is a sum of two squares (p. ix). The author in his preface (p. xviii) singles out as a typical example of this sort the theorem of Waring that every positive integer is the sum of a limited number of mth powers and gives a brief summary of the history of the theorem.

Of the many interesting discoveries in the theory of numbers announced by Fermat all have now been proved with the single exception of his so-called "last theorem," which states that it is impossible to separate any power higher than the second into a sum of two powers of the same degree. Concerning this theorem Fermat said: "I have discovered a truly remarkable proof which this margin is too small to contain." The final chapter of forty-six pages is devoted to the history of this theorem. "The dignity of this famous theorem was injured by the offer of a very large prize in 1908" (p. xix). "Fermat's last theorem is not of special importance in itself, and the publication of a complete proof would deprive it of its chief claim to attention for its own sake. But the theorem has acquired an important position in the history of mathematics on account of its

having afforded the inspiration which led Kummer to his invention of his ideal numbers, out of which grew the general theory of algebraic numbers, which is one of the most important branches of modern mathematics" (p. xix). Kummer's restoration of law in the midst of the chaos in the theory of algebraic numbers was one of the chief scientific triumphs of the last century.

Fermat's famous method of descent, infinite descent, indefinite descent, as it has been variously called by many writers, beginning with Fermat himself and continuing with his successors down to the author of the volume under review, comes in for mention in many places, as one may see from the subject index. The learner may make an interesting and profitable study of the method by means of these references to it. The only instance of a detailed proof left by Fermat is one by the method of descent; it is reproduced in full on pp. 615–616. Another very interesting case of the method is outlined on p. 619. [In the name of this method the adjective "infinite," and perhaps even the adjective "indefinite," is somewhat misleading. Is it desirable to adopt the practice which seems to predominate in the volume under review and call it simply the method of descent?]

Every fragmentary result connected with a question of importance suggests a problem for further investigation. These are too numerous in this volume for summary and are not of a nature to make this desirable. But there are a few conjectured or empirical or unproved theorems (besides the last one of Fermat) to which attention should be directed. We list the following:

1. Every number of the form $8k + 3$ is the sum of an odd square and the double of a prime $4n + 1$ (p. 261).

2. The triple of any odd square not divisible by 5 is a sum of squares of three primes other than 2 and 3 (p. 266).

3. The double of any odd integer is a sum of two primes $4n + 1$ (pp. 282, 289).

4. Every prime $18n \pm 1$ or else its triple is expressible in the form $x^3 - 3xy^2 \pm y^3$ (p. 575).

5. The sum of n numbers each a kth power is never a kth power if $n < k$ (pp. 648, 682). It seems to be unknown whether we can have $n = k$ when $k > 4$ (see pp. 682, 683).

6. See also pages 633, 752, 767.

Where the material to be gathered is so vast it is impossible that nothing has escaped attention. The author urges his readers to supply him with notices of errata or omissions as well as abstracts of the few papers marked by the symbol * before authors' names to signify that the papers were not available for report. The errata discovered by the reviewer will not give the reader trouble except possibly in the case of the error of Lexell on p. 732 in taking as relatively prime two factors which are not shown to have this property, so that the suggested proof is inadequate. We may here record a few facts which we have not found stated at those places in the volume at which it seemed natural to expect them:

1. Several persons treated the equation $x^3 + y^3 + z^3 = 2u^3$ (cf. pp. 563, 604) in *The Mathematical Visitor*, 2, 1887, pp. 84–88, one of whom, J. H. Drummond, gave the following identity:

$$\{24ab^5(1 + 6b^6)\}^3 + \{72ab^9(1 + 4b^6) + a\}^3 + \{72ab^9(1 + 4b^6) - a\}^3 = 2\{72ab^9(1 + 4b^6) + 6ab^3\}^3.$$

2. O. D. Kellogg has stated (Carmichael's *Diophantine Analysis*, p. 115; cf. Dickson, pp. 688–691) that for the positive integral solutions $x_1, x_2, \cdots, x_n$ of the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1$$

the maximum value of an x which can occur in a solution is u_n where $u_1 = 1$ and $u_{k+1} = u_k(u_k + 1)$. [It seems desirable to have a complete theory of this equation developed.]

3. For the equation $x^2y^2 + x^2 + y^2 = t^2$ Carmichael (*Diophantine Analysis*, p. 106) gave the solutions $x = a,\ y = 2a^2,\ t = a(2a^2 + 1)$; $x = a,\ y = 4a^3 + 4a^2 + 3a + 1,\ t = 4a^4 + 4a^3 + 5a^2 + 3a + 1$; and also certain solutions not in general integral for integral values of the parameters. He applied these to the solution of certain problems of Diophantus and Fermat.

4. C. Störmer (*Bull. Soc. France* 27, 1899, p. 160) showed that all the integral solutions k, m, n, x, y of the equation

$$m \arctan\frac{1}{x} + n \arctan\frac{1}{y} = k\,\frac{\pi}{4}$$

are the following: 1, 1, 1, 2, 3; 1, 2, $-$ 1, 2, 7; 1, 2, 1, 3, 7; 1, 4, $-$ 1, 5, 239.

It may be profitable to state for investigation a few (apparently unsolved or incompletely solved) problems some of which are perhaps of such sort as to be of interest to amateurs:

1. Determine all polynomial solutions u_a and v_a of the functional equation $a^2u_a^2 + a^2 + u_a^2 = v_a^2$ and apply the results to the solution of a group of Diophantine problems. [Suggestions for dealing with this and certain similar problems are given in the last chapter of my *Diophantine Analysis*.]

2. Develop the theory of the equation $x^4 + ay^4 + bu^4 + abv^4 = t^2$ for constant values of a and b.

3. Find the general integral solution of the equation $t^3 = x^3 + y^3 + 1$. [One solution is afforded by the relation $9^3 = 8^3 + 6^3 + 1$.]

4. Determine the properties of the integer m such that the equation $x^3 + y^3 + z^3 - 3xyz = mt^2$ shall have solutions and solve it for such values of m.

5. Determine the integral values of a for which the equation $x^4 + y^4 + a^2z^4 = u^4$ has non-zero integral solutions, and develop methods for finding these solutions.

On pages xx to xxi of the preface the author takes the reader into his confidence in a remarkable passage a part of which we shall quote. He had initially planned to give his work the title "topical history of the theory of numbers"; but the word topical was omitted on the advice of a prominent historian, since it is inconceivable that any one would desire the vast amount of material in this

work arranged otherwise than by topics. Having said this, the author then continues:

"Conventional histories take for granted that each fact has been discovered by a natural series of deductions from earlier facts and devote considerable space in the attempt to trace the sequence. But men experienced in research know that at least the germs of many important results are discovered by a sudden and mysterious intuition, perhaps the result of subconscious mental effort, even though such intuitions have to be subjected later to the sorting processes of the critical faculties. What is generally wanted is a full and correct statement of the facts, not an historian's personal explanation of those facts. The more completely the historian remains in the background or the less conscious the reader is of the historian's personality, the better the history. With such a view of the ideal self-effacement of the historian, what induced the author to interrupt his own investigations for the greater part of the past nine years to write this history? Because it fitted in with his conviction that every person should aim to perform at some time in his life some serious, useful work for which it is highly improbable that there will be any reward whatever other than his satisfaction therefrom."

It is refreshing and inspiring to find a man, when he pauses at a breathing place in the excellent performance of a great task, willing to set forth in a quiet way the fact that he has been moved by the highest and most unselfish ideal of duty.

R. D. CARMICHAEL.

Logarithmic and Trigonometric Tables. Revised edition. Prepared under the direction of E. R. HEDRICK. New York, Macmillan, 1920. 21 + 143 pp.

Preface: "The present edition of this book contains several tables not contained in the previous editions. The probability of the occurrence of errors has been minimized by using electrotype reproductions of the tables previously included, even when changes were made. Remarkably few errors existed in the original edition; what few have been discovered have been corrected.

"Minor changes only occur in the earlier pages. Care has been taken to preserve the page numbers of the principal tables up to page 114, so that older editions may be used in class-work without confusion, and texts which contain the principal tables may be used in the same class.

"Among the minor changes are the insertion of a condensed table of logarithms and anti-logarithms (Table Ia, p. 20), the insertion of a table of values of S and T for interpolation in logarithmic trigonometric functions (Table IIIa, p. 45), and the insertion on pages 1–19 of the logarithms of a few important numbers at appropriate points.

"The principal changes follow page 114. Tables VIII and IX (pp. 115–122) make reasonably complete the tables of hyperbolic functions formerly represented only by Table XII (pp. 112–114); These functions are of increasing importance, notably in Electrical Engineering.

"The table of haversines (Table X, pp. 123–125) will be welcomed particularly by those interested in navigation.

"The table of factors of composite numbers and logarithms of primes (Table XI, pp. 126–127) has obvious uses.

"Tables XII a, b, c, d, e, f, pages 128–132, are intended for work involving compound interest, annuities, depreciation, etc. They will be useful for statistics, insurance, accounting, and the mathematics of business.

"The same care has been exercised to eliminate errors in the new tables that resulted in so great a degree of reliability in the original edition of these tables."